University of California, Berkeley Physics 110B
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SHORT COURSE IN SPECIAL RELATIVITY

"To you it was fast."

1. Spacetime

According to Maxwell's equations, the speed c of light in vacuum is the same in all reference frames. Therefore c provides a fundamental link betweem distance and time. Instead of plotting time t in seconds, we can plot ct in meters. This motivates us to consider ct to be the fourth (or more conventionally the zeroth) component of a four-dimensional space called *spacetime*. A point ("event") in spacetime has the coordinates $r = (ct, \vec{r}) = (ct, x, y, z)$; r is called a *four-vector*.

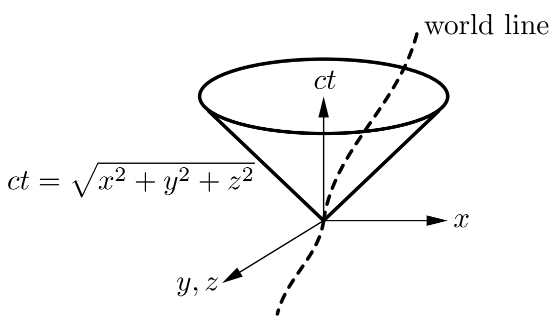

FIG. 1. Light cone of a particle and its world line, drawn in four spacetime dimensions using a randomly chosen Lorentz frame. (To simplify the sketch, the y and z axes are collapsed into a single direction.) The origin of coordinates is chosen to be the particle's position at $t = 0$. The world line's slope $d(ct)/ds$ (where $ds^2 \equiv dx^2 + dy^2 + dz^2$) everywhere must remain ≥ 1 so that the particle never exceeds the speed of light.

Choose a random inertial (Lorentz) frame and, at $t = 0$, define the space axes so that your own position is $x = y = z = 0$. Then consider your future. Since any information you create travels at most with the speed of light, only that part of spacetime with $c^2t^2 > x^2 + y^2 + z^2$ can possibly be affected by anything you are doing or will do. This is your *active future*, lying within the *light cone* sketched in Fig. 1. Your path through that future is your *world line*.

Likewise, a similar cone that points downward (not shown in the sketch) is your *active past*. It contains all the events that could possibly have affected you up to now. Apart from these cones, what remains is your *neutral region*. You are and have been unaware of any events in the neutral region, and, in turn, they will remain unaware of anything you are doing or will do.

If (in the absence of gravity) the universe consisted of a static four-dimensional sphere in spacetime centered on you (naturally), what fraction of spacetime's total volume would be active, *i.e.* would lie within your active light cones?

2. Distance in spacetime

Figure 2 shows a standard layout of two Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$, with relative ($\hat{x} = \hat{x}'$) velocity equal to $\beta_0 c$ (β_0 is dimensionless with the range $-1 \leq \beta_0 \leq 1$).

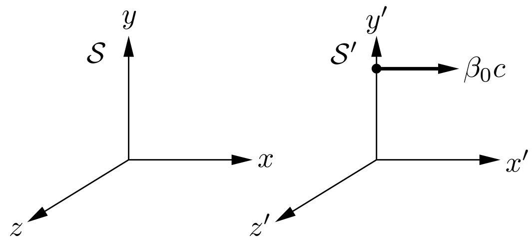

FIG. 2. Arrangement of two Lorentz frames $\mathcal{S}$ and $\mathcal{S}'$ to which the usual Lorentz transformation applies. With respect to frame $\mathcal{S}$, frame $\mathcal{S}'$ moves in the $\hat{x} = \hat{x}'$ direction with speed $\beta_0 c$. When the two 3D origins coincide, $t \equiv t' \equiv 0$.

Suppose that a pulse of electromagnetic (EM) radiation is emitted at $ct = ct' = 0$ when, according to Fig. 2, the 3D origins $x = y = z = 0$ and $x' = y' = z' = 0$ coincide. In either frame, Maxwell's equations force this pulse to be a spherical bubble expanding from the 3D origin with the speed of light:

$$x^2 + y^2 + z^2 = c^2 t^2$$
$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \ .$$

For this EM bubble, it is definitely true that

$$c^2 t^2 - x^2 - y^2 - z^2 = c^2 t'^2 - x'^2 - y'^2 - z'^2 \quad (1)$$

Keeping this result in mind, we consider how to define the length of a spacetime four-vector r that extends from the 4D origin to (ct, x, y, z). If we hadn't already analyzed the EM bubble, perhaps our first thought would be to proceed by analogy to the length^2 of a vector in three spatial dimensions:

$$\vec{r} \cdot \vec{r} = x^2 + y^2 + z^2$$
$$r \cdot r =?\ c^2 t^2 + \vec{r} \cdot \vec{r} \ .$$

But we want the length of a spacetime four-vector to remain invariant to the choice of Lorentz frame (much as a 3D vector's length is invariant to the choice of 3D coordinate orientation). Accepting this requirement, we are forced by Eq. (1) to change the sign of the last term:

$$r \cdot r = c^2 t^2 - \vec{r} \cdot \vec{r} \quad (2)$$

By extension, the inner product of two four-vectors r_A and r_B is

$$r_A \cdot r_B = ct_A\, ct_B - x_A x_B - y_A y_B - z_A z_B \quad (3)$$

[Some authors (*i.e.* Griffiths) instead define an inner product of opposite sign, but most authors and physicists use Eq. (3)'s convention.]

Obviously from Eq. (2), $r \cdot r$ can be negative (strange for a length^2!) as well as positive. So can the interval^2 $\Delta r \cdot \Delta r \equiv (r_A - r_B) \cdot (r_A - r_B)$ between two spacetime events r_A and r_B. Such intervals are called

$$\textit{timelike} \text{ if } \Delta r \cdot \Delta r > 0 \quad (c^2(\Delta t)^2 > \Delta\vec{r} \cdot \Delta\vec{r})$$
$$\textit{lightlike} \text{ if } \Delta r \cdot \Delta r = 0 \quad (c^2(\Delta t)^2 = \Delta\vec{r} \cdot \Delta\vec{r})$$
$$\textit{spacelike} \text{ if } \Delta r \cdot \Delta r < 0 \quad (c^2(\Delta t)^2 < \Delta\vec{r} \cdot \Delta\vec{r}) \ .$$

Because the inner product is invariant to the choice of Lorentz frame, so is the time-, light-, or space-likeness of the interval between any pair of events.

Except for effects of quantum entanglement, pairs of events can be causally connected only if the interval between them is timelike (*within* the light cone) or lightlike (*on* the light cone). Event A can't cause event B if the two events are separated by a spacelike interval; on the contrary, a Lorentz frame could be found in which A and B are simultaneous, or, worse yet, occur in reverse order!

3. Infinitesimal rotation in space

We can understand more about transformations in spacetime by reviewing the properties of ordinary space rotations. Figure 3 sketches the geometry appropriate for a passive (coordinate-system) rotation in 2D space.

In this section we assume that the rotation is *infinitesimal* ($\phi \ll 1$). Then, from the figure,

$$x' = x + \phi y$$
$$y' = -\phi x + y \ ,$$

or, in matrix notation,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \phi \\ -\phi & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (4)$$

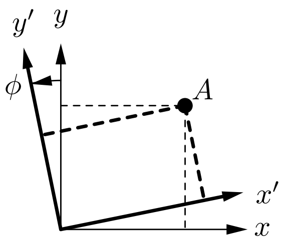

FIG. 3. Passive rotation in two Euclidean dimensions. Point A, which is not *actively rotated, may be expressed either as* (x, y) *or as* (x', y')*. Relative to the unprimed frame, the primed frame is rotated by the positive (counterclockwise) angle* ϕ*. 2D rotations preserve* $x'^2 + y'^2 = x^2 + y^2$.

The distance[2] between point A and the origin is

$$\begin{aligned} \vec{r} \cdot \vec{r} &= x^2 + y^2 \\ \vec{r}\,' \cdot \vec{r}\,' &= x'^2 + y'^2 \\ &= (x + \phi y)^2 + (y - \phi x)^2 \\ &= x^2 + y^2 + 2\phi xy - 2\phi xy + \phi^2(x^2 + y^2) \\ &= (x^2 + y^2)(1 + \phi^2)\ . \end{aligned}$$

As expected, this distance is the same in the primed and unprimed coordinate systems, provided that we are willing to ignore ϕ^2 compared to 1. This is reasonable, since ϕ^2 is second order in the infinitesimal quantity ϕ. If we were concerned about this term, we could rewrite Eq. (4) as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{1+\phi^2}} \begin{pmatrix} 1 & \phi \\ -\phi & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

Then the distance between point A and the origin would be exactly the same in the two systems.

3. Infinitesimal transformation in spacetime

Figure 4 sketches the geometry appropriate for a passive (coordinate-system) transformation in 2D spacetime. There $r = (ct, x)$ and $r' = (ct', x')$ are the coordinates of event A as viewed in $\mathcal{S}$ and $\mathcal{S}'$, respectively. Temporarily, we denote the transformation parameter by β_0. In this section we assume that the transformation is *infinitesimal* ($\beta_0 \ll 1$). Then, from the figure,

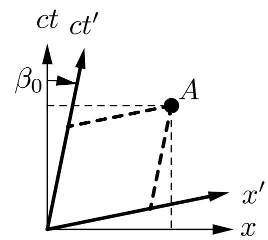

FIG. 4. Infinitesimal passive transformation in two spacetime dimensions. Event A, which is not *actively transformed, may be expressed either as* (ct, x) *or as* (ct', x')*. Relative to the unprimed frame, the primed frame is arranged as in Fig. 2. 2D Lorentz transformations preserve* $ct'^2 - x'^2 = ct^2 - x^2$.

$$\begin{aligned} ct' &= ct - \beta_0 x \\ x' &= -\beta_0 ct + x\ , \end{aligned}$$

or, in matrix notation,

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & -\beta_0 \\ -\beta_0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{6}$$

Using Eq. (2), the spacetime interval[2] between event A and the origin is

$$\begin{aligned} r \cdot r &= c^2t^2 - x^2 \\ r' \cdot r' &= c^2t'^2 - x'^2 \\ &= (ct - \beta_0 x)^2 - (x - \beta_0 ct)^2 \\ &= \ c^2t^2 - x^2 - 2\beta_0 ctx + 2\beta_0 ctx - \\ &- \beta_0^2(c^2t^2 - x^2) \\ &= (c^2t^2 - x^2)(1 - \beta_0^2)\ . \end{aligned}$$

Why did we draw Fig. 4 in this peculiar way? We did so because we needed a minus sign in the top right-hand element of the 2×2 matrix in Eq. (6). With the help of this minus sign, we were able to force $r \cdot r$ to be equal to $r' \cdot r'$, provided, again, that we are willing to ignore a term that is second order in the infinitesimal parameter β_0. If we were concerned about this term, we could rewrite Eq. (6) as

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1-\beta_0^2}} \begin{pmatrix} 1 & -\beta_0 \\ -\beta_0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{7}$$

Then the interval between event A and the origin would be exactly the same in the two systems.

When $-\beta_0 x$ is ignored with respect to ct in Eq. (6), we recover the *Galilei transformation* that you used in high school to solve distance = rate × time problems:

$$\begin{aligned} t' &\approx t \\ x' &= x - \beta_0 ct = x - Vt \ , \end{aligned} \tag{8}$$

where V is the relative velocity between two slow coordinate systems (*e.g.* trains). Requiring agreement with the Galilei transformation in this limit forces β_0 to be equal to V/c – there is no other viable choice.

4. Finite rotation in space

For ordinary rotations in 2D Euclidean space, when the rotation angle ϕ is not necessarily $\ll 1$, Eq. (4) takes the familiar form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{9}$$

Why do we choose the functions $\cos\phi$ and $\sin\phi$? Most directly, we apply trigonometry to Fig. 3. However, we could merely have searched for two functions $C(\phi)$ and $S(\phi)$ which approach unity and ϕ, respectively, as $\phi \to 0$ – and which satisfy the property $C^2(\phi) + S^2(\phi) = 1$ for any ϕ. This property guarantees that

$$\vec{r} \cdot \vec{r} = \vec{r}\,' \cdot \vec{r}\,'$$

for any ϕ, preserving the lengths of vectors after any coordinate rotation. The choices $C(\phi) = \cos\phi$ and $S(\phi) = \sin\phi$ satisfy those requirements.

5. Finite transformation in spacetime

For infinitesimal spacetime transformations, we used the transformation parameter $\beta_0 = V/c$ to agree with Galilei. Now, when the spacetime transformation is no longer infinitesimal, we reexamine this choice.

What properties should the transformation parameter have? For rotations in Euclidean space, the accepted parameter is the rotation angle ϕ. It has the property of being *additive*: two successive rotations about the same axis by angles ϕ_1 and ϕ_2 are equivalent to one rotation by $\phi_1 + \phi_2$. In spacetime, it's clear that β cannot be additive; if it were, a sequence of transformations each with $\beta_i < 1$ would yield $\beta_{\text{tot}} > 1$, exceeding the speed of light.

We shall call the additive parameter for spacetime transformations η, the *boost*. So far, all we know about η is that it is a function of β which approaches β in the slow ($\beta \to 0$) limit.

By analogy with section 4, we generalize Eq. (6) to finite spacetime transformations:

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} C(\eta_0) & -S(\eta_0) \\ -S(\eta_0) & C(\eta_0) \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{10}$$

Again $C(\eta_0)$ and $S(\eta_0)$ are two (as yet unspecified) functions of the (as yet unspecified) transformation parameter η_0. Using Eq. (2) as we did in section 3, the spacetime interval^2 between event A and the origin is

$$\begin{aligned} r \cdot r &= c^2t^2 - x^2 \\ r' \cdot r' &= c^2t'^2 - x'^2 \\ &= (ctC - xS)^2 - (xC - ctS)^2 \\ &= \ c^2t^2C^2 - x^2C^2 - 2ctxCS + \\ &+ 2ctxCS - (c^2t^2S^2 - x^2S^2) \\ &= (c^2t^2 - x^2)\big(C^2(\eta_0) - S^2(\eta_0)\big) \ . \end{aligned}$$

Evidently, to preserve the spacetime interval^2 after a finite transformation, the matrix elements $C(\eta_0)$ and $S(\eta_0)$ must satisfy

$$C^2(\eta_0) - S^2(\eta_0) = 1 \ .$$

When $\eta_0 \ll 1$, we know that η_0 approaches β_0. Comparing Eq. (10) with Eq. (6), it's clear as well that $S(\eta_0)$ must approach η_0 and and $C(\eta_0)$ must approach unity in this limit.

The functions that satisfy these requirements are

$$\begin{aligned} C(\eta_0) &= \cosh\eta_0 \equiv \frac{e^{\eta_0} + e^{-\eta_0}}{2} \\ S(\eta_0) &= \sinh\eta_0 \equiv \frac{e^{\eta_0} - e^{-\eta_0}}{2} \end{aligned} \tag{11}$$

where cosh and sinh are the *hyperbolic cosine* and *hyperbolic sine*, respectively. From their definitions, it's clear that $C^2 - S^2 = 1$. Expanding

$$e_0^\eta \approx 1 + \eta_0 + \tfrac{1}{2}\eta_0^2 + \dots \,,$$

it's easy to confirm that $\sinh\eta_0$ approaches η_0 and $\cosh\eta_0$ approaches unity when $\eta_0 \ll 1$, as we require. Substituting these hyperbolic functions in Eq. (10), the finite spacetime transformation begins to resemble the ordinary Euclidean rotation in Eq. (9):

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\eta_0 & -\sinh\eta_0 \\ -\sinh\eta_0 & \cosh\eta_0 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{12}$$

To learn more about the boost parameter η_0, we rearrange this equation using the *hyperbolic tangent*

$$\tanh\eta_0 \equiv \frac{\sinh\eta_0}{\cosh\eta_0} = \frac{e^{\eta_0} - e^{-\eta_0}}{e^{\eta_0} + e^{-\eta_0}}$$

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \cosh\eta_0 \begin{pmatrix} 1 & -\tanh\eta_0 \\ -\tanh\eta_0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Using the identity

$$\begin{aligned} \cosh\eta_0 &= \sqrt{\cosh^2\eta_0} \\ &= \sqrt{\frac{\cosh^2\eta_0}{\cosh^2\eta_0 - \sinh^2\eta_0}} \\ &= \sqrt{\frac{1}{1-\tanh^2\eta_0}}\,, \end{aligned} \tag{13}$$

Eq. (12) takes the form

$$\begin{aligned} \begin{pmatrix} ct' \\ x' \end{pmatrix} &= \sqrt{\frac{1}{1-\tanh^2\eta_0}} \times \\ &\times \begin{pmatrix} 1 & -\tanh\eta_0 \\ -\tanh\eta_0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} . \end{aligned}$$

Comparing this with Eq. (7), we identify

$$\begin{aligned} &\beta_0 = \tanh\eta_0 \;\; \text{or} \\ &\eta_0 = \tanh^{-1}\beta_0 = \tanh^{-1}\left(\frac{V}{c}\right) \end{aligned} \tag{14}$$

We have learned to equate the *boost* η_0 – the additive parameter for spacetime transformations – to the *arc hyperbolic tangent* of $\beta_0 \equiv V/c$. Though we can add many boosts to make $|\eta_0|$ arbitrarily large, $|V|$ never exceeds c because $|\tanh\eta_0|$ never exceeds unity.

6. Lorentz transformation

Equations (12) and (14) together define the *Lorentz transformation* in its basic form. The matrix elements in Eq. (12) are functions of the fundamental additive parameter η_0 defined in Eq. (14); these functions show an intimate relation to the circular functions used for ordinary rotations in Euclidean space.

For solving problems involving only one spacetime transformation, without any acceleration, a more convenient form for the Lorentz transformation is

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma_0 & -\gamma_0\beta_0 \\ -\gamma_0\beta_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} \tag{15}$$

where, using Eq. (13),

$$\gamma_0 \equiv \cosh\eta_0 = \frac{1}{\sqrt{1-\beta_0^2}} \tag{16}$$

In many introductory texts, which seek to avoid matrices and Greek letters, Eq. (15) is written

$$\begin{aligned} t' &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left(t - \frac{V}{c^2}x\right) \\ x' &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\left(x - Vt\right) \end{aligned} \tag{17}$$

Equations (12), (15), and (17) equivalently transform ct and x from inertial frame $\mathcal{S}$ to inertial frame $\mathcal{S}'$, where $\mathcal{S}'$ moves in the $\hat{x} = \hat{x}'$ direction relative to $\mathcal{S}$ with velocity $V = \beta_0 c$ as in Fig. 2. How would we transform instead from $\mathcal{S}'$ to $\mathcal{S}$? The only feature that distinguishes $\mathcal{S}'$ from $\mathcal{S}$ is the fact that $\beta_0 c$ is the $\hat{x} = \hat{x}'$ velocity of $\mathcal{S}'$ *relative* to $\mathcal{S}$; conversely, the velocity of $\mathcal{S}$ relative to $\mathcal{S}'$ is $-\beta_0 c$. Therefore the *inverse* Lorentz transformation is the same as the *direct* transformation with the sign of β_0 reversed:

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma_0 & +\gamma_0\beta_0 \\ +\gamma_0\beta_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix} \tag{18}$$

A bit of algebra will confirm that the direct Lorentz transformation followed by its inverse leaves us back where we started:

$$\begin{pmatrix} \gamma_0 & \gamma_0\beta_0 \\ \gamma_0\beta_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} \gamma_0 & -\gamma_0\beta_0 \\ -\gamma_0\beta_0 & \gamma_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

7. Lorentz transformation in 4 dimensions

When the velocity $\vec{\beta}_0 c$ of $\mathcal{S}'$ relative to $\mathcal{S}$ is in the $\hat{x} = \hat{x}'$ direction, as in Fig. 2, the Lorentz transformation doesn't change the y and z coordinates:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma_0 & -\gamma_0\beta_0 & 0 & 0 \\ -\gamma_0\beta_0 & \gamma_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{19}$$

(Up to now we omitted the 3$^{\text{rd}}$ and 4$^{\text{th}}$ dimensions to save space.) This can be written in matrix notation as

$$r' = \Lambda r \tag{20}$$

where r' and r are the 4×1 column vectors and Λ is the 4×4 transformation matrix. More generally, if $\vec{\beta}_0$ is in an arbitrary direction $\hat{n}$, Eq. (20) becomes

$$r' = \Lambda_R^{-1} \Lambda \Lambda_R \, r \tag{21}$$

where Λ_R is a matrix that performs the 3D spatial rotation which transforms the $\hat{n}$ direction into the $\hat{x}$ direction. In general, Λ_R takes the form

$$\Lambda_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_{xx} & \lambda_{xy} & \lambda_{xz} \\ 0 & \lambda_{yx} & \lambda_{yy} & \lambda_{yz} \\ 0 & \lambda_{zx} & \lambda_{zy} & \lambda_{zz} \end{pmatrix} ,$$

and has the *orthogonality* property

$$\left(\Lambda_R^{-1}\right)_{ij} = \left(\Lambda_R\right)_{ji} .$$

8. Time dilation

Figure 5 shows a clock, attached to $\mathcal{S}'$, that ticks at times t'_1 and t'_2. As observed in $\mathcal{S}$, which is *not* at rest with respect to the clock, what time interval $t_2 - t_1$ separates these ticks?

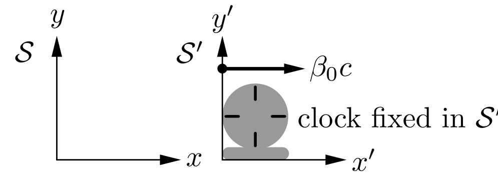

FIG. 5. Observing time dilation. Frames $\mathcal{S}$ and $\mathcal{S}'$ are arranged as in Fig. 2. A clock is fixed to $\mathcal{S}'$, which is the proper frame *because two spacetime events (clock ticks) whose time separation is of interest occur in the same* place *in that frame. In any other Lorentz frame, the time interval between these ticks can be measured with a fine grid of clocks, rulers, and data loggers, avoiding any observational errors due to signal propagation. This time interval is larger (dilated) than the one observed in the proper frame.*

Applying the *inverse* Lorentz transformation,

$$\begin{aligned} ct_2 &= \gamma_0 ct'_2 + \gamma_0\beta_0 x'_2 \\ ct_1 &= \gamma_0 ct'_1 + \gamma_0\beta_0 x'_1 . \end{aligned}$$

Now $x'_2 = x'_1$ because, as seen in $\mathcal{S}'$, the clock is always in the same position. Subtracting the second equation from the first,

$$\begin{aligned} c(t_2 - t_1) &= \gamma_0 c(t'_2 - t'_1) \\ \Delta t &= \gamma_0 \Delta t' \equiv \gamma_0 \Delta\tau \end{aligned} \tag{22}$$

Since γ_0 is always ≥ 1, the time interval between ticks is longer in frame $\mathcal{S}$, which is moving with respect to the *unique* frame $\mathcal{S}'$, where the ticks occur at the same place. Since $\mathcal{S}'$ is unique, in Eq. (22) we assigned a unique name $\Delta\tau$ to the time interval $\Delta t'$ observed in this frame. $\mathcal{S}'$ is called the *proper frame* and τ is called the *proper time.*

At the expense of slightly more algebra, the same result also could be obtained from the direct Lorentz transformation.

9. Space contraction

Figure 6 shows a rod attached to $\mathcal{S}'$. When measured at any time in that frame, its ends are at x'_1 and x'_2. As observed at the same time $t_1 = t_2$ in $\mathcal{S}$, which is *not* at rest with respect to the rod, what distance $x_2 - x_1$ separates the rod ends?

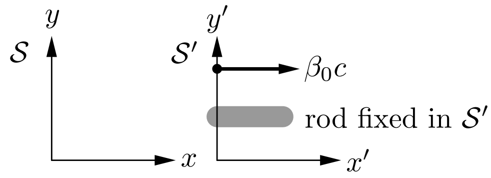

FIG. 6. Observing space contraction. Frames $\mathcal{S}$ and $\mathcal{S}'$ are arranged as in Fig. 2. A rod is fixed to $\mathcal{S}'$. In any other Lorentz frame, the positions of these ends can be measured simultaneously using a fine grid of clocks, rulers, and data loggers, avoiding any observational errors due to signal propagation. There the distance between the ends is smaller (contracted) than in $\mathcal{S}'$.

Applying the *direct* Lorentz transformation,

$$\begin{aligned} x'_2 &= \gamma_0 x_2 - \gamma_0 \beta_0 c t_2 \\ x'_1 &= \gamma_0 x_1 - \gamma_0 \beta_0 c t_1 \ . \end{aligned}$$

Using the fact that $t_2 = t_1$, and subtracting the second equation from the first,

$$\begin{aligned} x'_2 - x'_1 &= \gamma_0 (x_2 - x_1) \\ \Delta x &= \frac{\Delta x'}{\gamma_0} \end{aligned} \tag{23}$$

Since γ_0 is always ≥ 1, the rod appears shorter in frame $\mathcal{S}$, which is moving with respect to the unique frame $\mathcal{S}'$ to which the rod is attached.

10. Velocity addition

This classic problem is outlined in the caption to Fig. 7. Clearly two velocities cannot simply add – otherwise the sum could exceed the velocity of light. What is the *Einstein law of velocity addition*?

The problem is solved most elegantly by use of the boost parameter η, because it is additive:

$$\begin{aligned} \eta'' &= \eta + \eta' \\ \beta'' &= \tanh \eta'' \\ &= \tanh (\eta + \eta') \ . \end{aligned}$$

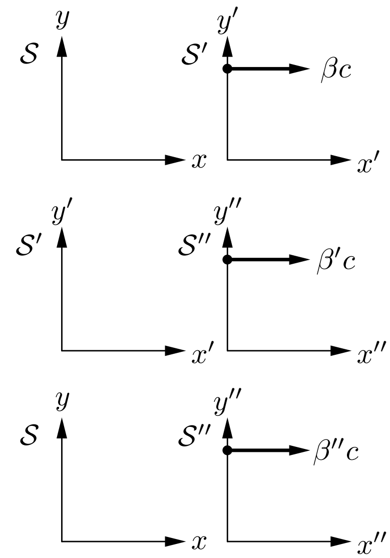

FIG. 7. Arrangement for adding relativistic velocities sharing a common direction. Frame $\mathcal{S}'$ moves in the $\hat{x} = \hat{x}'$ direction at velocity βc with respect to frame $\mathcal{S}$. Frame $\mathcal{S}''$ moves in the $\hat{x}' = \hat{x}''$ direction at velocity $\beta' c$ with respect to frame $\mathcal{S}'$. With what velocity $\beta'' c$ does frame $\mathcal{S}''$ move with respect to $\mathcal{S}$?

The identity

$$\tanh (a + b) = \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b}$$

can be taken on faith, in analogy to the more familiar

$$\tan (a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} \ ,$$

or it can be derived in a few lines starting from the definition of the hyperbolic tangent. Using this identity,

$$\begin{aligned} \beta'' &= \frac{\tanh \eta + \tanh \eta'}{1 + \tanh \eta \tanh \eta'} \\ &= \frac{\beta + \beta'}{1 + \beta \beta'} \ . \end{aligned} \tag{24}$$

The combined β'' never exceeds unity.

11. Human constraints on space travel

Assume that an astronaut is willing to be accelerated at no more than 1 g, and to age no more than 40 years during the voyage. What maximum velocity can be achieved? How far will the astronaut travel, and how much time will have elapsed on earth?

The full voyage consists of $\tau_{10} \equiv 10$ years with acceleration $a'_x = +g$, 20 years with $a'_x = -g$, and 10 years with $a'_x = +g$. We need consider only the first leg. To answer the questions posed, we'll double the first-leg distance and quadruple the first-leg time.

Because the astronaut is accelerating, his/her rest frame is *not* inertial. However, to analyze his/her motion using the Lorentz transformation, we need an inertial frame. Accordingly we define a *comoving frame* $\mathcal{S}'$ which at a certain moment is at rest with respect to the astronaut but which is *not* accelerating. Then, with respect to the comoving frame, the astronaut moves with velocity $\beta_{\rm rel}c$, where $\beta_{\rm rel} = 0$ at a certain moment of astronaut time τ.

Next allow an infinitesimal unit $d\tau$ of astronaut time to elapse. As seen in the comoving frame $\mathcal{S}'$, the elapsed time dt' is the same as $d\tau$ because the two frames are still moving only infinitesimally slowly ($\beta_{\rm rel} \ll 1$) with respect to each other. Likewise, because $\beta_{\rm rel} \ll 1$, the acceleration felt by the astronaut is the same as the acceleration observed in $\mathcal{S}'$. Therefore, after astronaut time interval $d\tau$ has elapsed, the astronaut appears in $\mathcal{S}'$ to be moving with relative velocity $c\, d\beta_{\rm rel} = g\, d\tau$.

To sum up these increments, we need to use the boost parameter η, which is additive. As seen in Earth frame $\mathcal{S}$, each incremental boost $d\eta$, calculated in a different comoving frame, will add linearly to yield the total boost. Fortunately, when $\beta_{\rm rel} \ll 1$, $d\eta = d\beta_{\rm rel}$. Therefore our working equation is

$$d\eta = \frac{g}{c}\, d\tau \; .$$

Integrating,

$$\begin{aligned}
\eta_{\rm max} &= \int_0^{\tau_{10}} \frac{g}{c}\, d\tau \\
&= \frac{g}{c}\, \tau_{10} \\
&= 10.34 \\
\beta_{\rm max} &= \tanh \eta_{\rm max} \\
&= 1 - (2.09 \times 10^{-9}) \; .
\end{aligned}$$

The distance covered is obtained by integrating astronaut displacements as observed in the Earth's frame:

$$\begin{aligned}
dx &= \beta c\, dt \\
&= (\tanh \eta) c\, \gamma d\tau \quad \text{(time dilation)} \\
&= c\, (\tanh \eta \cosh \eta)\, d\tau \\
&= c\, (\sinh \eta)\, d\tau \\
\Delta x &= (2 \text{ legs}) \times \int_0^{\tau_{10}} c\, (\sinh \eta)\, d\tau \\
&= 2c \int_0^{\tau_{10}} \sinh \left(\frac{g\tau}{c}\right) d\tau \\
&= 2\frac{c^2}{g} \left(\cosh \left(\frac{g}{c}\, \tau_{10}\right) - 1\right) \\
&= 2.84 \times 10^{20} \text{ m} \\
&= 29,900 \text{ light yr} \; .
\end{aligned}$$

Considering that the universe has been flying apart at nearly the speed of light for ≈14 billion years since the Big Bang, it's clear that only an infintesimal fraction of it can be explored within an astronaut's lifetime.

Finally, the time elapsed on earth is:

$$\begin{aligned}
dt &= \gamma\, d\tau \quad \text{(time dilation)} \\
&= \cosh \eta\, d\tau \\
\Delta t &= (4 \text{ legs}) \times \int_0^{\tau_{10}} (\cosh \eta)\, d\tau \\
&= 4 \int_0^{\tau_{10}} \cosh \left(\frac{g\tau}{c}\right) d\tau \\
&= 4\frac{c}{g} \sinh \left(\frac{g}{c}\, \tau_{10}\right) \\
&= 1.89 \times 10^{12} \text{ sec} \\
&= 59,850 \text{ yr} \quad \text{(compare 40 yr!)} \; .
\end{aligned}$$

This last result is often called the "twin paradox" because the earthbound twin ages more rapidly (1500× more rapidly in this example). It *isn't* a paradox, because the astronaut twin, who is accelerating, is fundamentally different from the earthbound twin, who isn't.

12. Four-momentum

In Eq. (3) we saw that the inner product $r_A \cdot r_B$ of two spacetime four-vectors remains the same after a Lorentz transformation. It is called a *Lorentz invariant*.

An interval of proper time $d\tau$ and a particle's rest-frame mass m are also Lorentz invariants. This is a trivial statement: to determine $d\tau$ or m, an observer in an arbitrary inertial frame must transform to a different frame (the proper frame to get $d\tau$, or the particle's rest frame to get m). If all observers in all their individual inertial frames are able to perform these transformations, they will all agree on $d\tau$ and m.

The *four-momentum* p is defined by

$$\begin{aligned} p &\equiv m\frac{dr}{d\tau} \\ &= \big(mc\frac{dt}{d\tau}, m\frac{dx}{d\tau}, m\frac{dy}{d\tau}, m\frac{dz}{d\tau}\big) \\ dt &= \gamma\, d\tau \quad \text{(time dilation)} \\ p &= \big(\gamma mc, \gamma m\frac{dx}{dt}, \gamma m\frac{dy}{dt}, \gamma m\frac{dz}{dt}\big) \\ &= \big(\gamma mc, \gamma m\vec{v}\big) \\ &\equiv \big(\frac{E}{c}, \vec{p}\big) \ . \end{aligned} \tag{25}$$

Note that Eq. (25) defines the *total energy* E and the *relativistic momentum* $\vec{p}$:

$$\begin{aligned} p_0 = \gamma mc &\equiv \frac{E}{c} \\ (p_x, p_y, p_z) = \gamma m\vec{v} &\equiv \vec{p}\ . \end{aligned} \tag{26}$$

Because m and $d\tau$ are Lorentz invariants, the four-momentum p transforms in the same way as the spacetime coordinate r:

$$\begin{pmatrix} E'/c \\ p'_x \end{pmatrix} = \begin{pmatrix} \gamma_0 & -\gamma_0\beta_0 \\ -\gamma_0\beta_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} E/c \\ p_x \end{pmatrix} \tag{27}$$

Therefore p is also a *four-vector*.

Correspondingly, the length^2 $p \cdot p$ of the four-momentum is a Lorentz invariant:

$$\begin{aligned} p \cdot p &= (\gamma mc, \gamma m\vec{v}) \cdot (\gamma mc, \gamma m\vec{v}) \\ &= \gamma^2 m^2 (c^2 - v^2) \\ &= \gamma^2 m^2 c^2 (1 - \beta^2) \\ &= m^2 c^2\ . \end{aligned}$$

(This result can also be obtained by evaluating $p \cdot p$ in the particle's rest frame, where $\gamma = 1$ and $\vec{v} = 0$.) Thus the *basic equation for solving relativistic kinematics problems* is

$$p \cdot p = \frac{E^2}{c^2} - \vec{p} \cdot \vec{p} = m^2 c^2 \tag{28}$$

We've called E the "total energy", but we haven't yet related it to any other energy. Making a Taylor series expansion,

$$\begin{aligned} E &= \gamma mc^2 \\ &= \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \\ &= mc^2\big(1 + \tfrac{1}{2}\frac{v^2}{c^2} + \ldots\big) \\ &= mc^2 + \tfrac{1}{2}mv^2 + \ldots \\ &\equiv mc^2 + T\ . \end{aligned} \tag{29}$$

The total energy E is equal to the *rest mass energy* mc^2 plus the *relativistic kinetic energy* $T \equiv E - mc^2$. T is *not* equal to $\frac{1}{2}mv^2$ – this is true only in the nonrelativistic limit, when the extra terms in Eq. (29) can be dropped.

Because $c^2 \approx (3 \times 10^8\ \text{m})^2$ is large, we recognize the possibility of converting mass to *lots* of energy.

13. Compton scattering

To illustrate the power of Eq. (28) for solving problems in relativistic kinematics, we consider the scattering of a quantum of light (a massless *photon*) by an electron at rest. Following A.H. Compton, we seek a relation between the photon's scattering angle θ and its loss of energy as a result of the scatter.

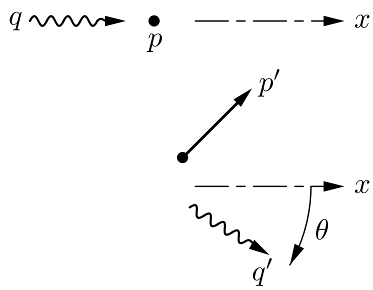

FIG. 8. Geometry for photon-electron ("Compton") scattering. The incident and scattered photon four-momenta are denoted by q and q', respectively, and the target- and recoil-electron four-momenta are denoted by p and p'. The incident photon energy is known and the scattered photon energy is measured, as is the photon's scattering angle θ. The target electron is assumed to be (essentially) at rest; the recoil electron is unobserved.

The four-momenta of the participants in this reaction are defined in Fig. 8. Because the incident photon is travelling in the x direction, its four-momentum can be written $q = (q_0, q_x, 0, 0)$. Since photons are massless, $q \cdot q = m^2c^2 = 0$. Therefore $q_x = q_0$. Defining the y axis so that the scattering takes place in the xy plane, for the incident and scattered photon four-momenta and the target electron four-momentum we can write

$$\begin{aligned} q &= (q_0, q_0, 0, 0) \\ q' &= (q'_0, q'_0 \cos\theta, q'_0 \sin\theta, 0) \\ p &= (mc, 0, 0, 0) \ , \end{aligned}$$

where m is the electron's rest mass.

In any scattering process, due to invariance of physical laws with respect to coordinate displacements both in position and in time, both (relativistic) momentum and (total) energy are conserved. (Recall that *kinetic* energy is conserved only in *elastic* collisions.) Energy and momentum conservation can be expressed as four equations

$$\begin{aligned} q_0 + p_0 &= q'_0 + p'_0 \\ q_x + p_x &= q'_x + p'_x \\ q_y + p_y &= q'_y + p'_y \\ q_z + p_z &= q'_z + p'_z \end{aligned}$$

or as a single four-vector equation

$$q + p = q' + p' \ .$$

The latter is more elegant. Rearranging and squaring it,

$$\begin{aligned} p' &= p + q - q' \\ m^2c^2 &= [p + (q - q')] \cdot [p + (q - q')] \\ &= m^2c^2 + 2p \cdot (q - q') + (q - q') \cdot (q - q') \\ 0 &= 2p \cdot (q - q') + q \cdot q + q' \cdot q' - 2q \cdot q' \\ &= 2p \cdot (q - q') + 0 + 0 - 2q \cdot q' \\ &= p \cdot (q - q') - q \cdot q' \\ &= mc(q_0 - q'_0) - q_0 q'_0 + q_0 q'_0 \cos\theta \\ &= \frac{q_0 - q'_0}{q_0 q'_0} - \frac{1 - \cos\theta}{mc} \\ &= \frac{1}{q'_0} - \frac{1}{q_0} - \frac{1 - \cos\theta}{mc} \ . \end{aligned}$$

Usually this result is multiplied by *Planck's constant* h, with the photon wavelength λ equal to h/q_0. Then

$$\lambda' - \lambda = \lambda_C (1 - \cos\theta) \tag{30}$$

where λ_C is the *Compton wavelength* of the electron, equal to

$$\lambda_C = \frac{h}{mc} = 2\pi \times 386 \times 10^{-15} \text{ m.}$$

Planck's constant is

$$h = 2\pi \times 6.58 \times 10^{-16} \text{ eV sec.}$$

14. Propulsion constraints on space travel

In section 11 we found that an astronaut who is willing to travel for 40 years while experiencing an acceleration of 1 g can cover only a paltry 29,900 light years (and back). Seems like a modest goal – but are we able to design a rocket engine that would accomplish even that much?

Again we work in a comoving frame (Fig. 9), instantaneously at rest relative to the rocket at $\tau = \tau_0$. In an infinitesimal proper time interval $d\tau$, the rocket ejects what we'll call "particle

#1", with energy dE_1 and velocity $\vec{\beta}_1 c$ relative to the comoving frame. The rocket loses an amount of mass dm (defined positive) and recoils with infinitesimal velocity $d\vec{\beta}$.

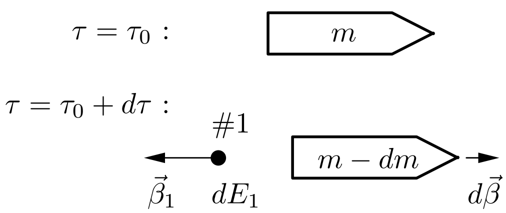

FIG. 9. Analysis of spacecraft propulsion in the comoving frame, *an inertial frame instantaneously at rest with respect to the spacecraft at the spacecraft's proper time τ_0. At $\tau = \tau_0$ in this frame, the spacecraft appears to be at rest, though it is accelerating. At $\tau = \tau_0 + d\tau$, due to ejection of a (positive) infinitesimal mass dm at relative velocity $\vec{\beta}_1$, the spacecraft has acquired a velocity $d\vec{\beta}$. Because $d\vec{\beta}$ is $\ll 1$, the spacecraft's proper time τ is still equivalent to the time measured in the comoving frame.*

As observed in the comoving frame, define the rocket four-momentum to be P_0 at $\tau = \tau_0$, and P' at time $\tau = \tau_0 + d\tau$; define the four-momentum of particle #1 to be p_1 at the later time. Then

$$\begin{aligned} P_0 &= (mc, \vec{0}) \\ P' &\approx \big((m - dm)c + \tfrac{1}{2}mc|d\vec{\beta}|^2, mc\, d\vec{\beta}\big) \\ p_1 &= \big(\frac{dE_1}{c}, \vec{\beta}_1 \frac{dE_1}{c}\big) \ . \end{aligned}$$

In assigning the components of p_1, we made use of the relation $\vec{p} = \vec{\beta}E/c$, which follows from the definition of the four-momentum. In assigning the components of P', we took advantage of the fact that, in the comoving frame, the rocket is still nonrelativistic at $\tau = \tau_0 + d\tau$, so that E is approximately equal to $\frac{1}{2}mv^2$ plus the rest mass energy.

If we assume that the rocket engine is perfectly efficient, so that no heat energy is radiated in random directions, energy and momentum conservation require that

$$P_0 = p_1 + P' \ .$$

We separate this equation into a timelike part

$$mc = \frac{dE_1}{c} + (m - dm)c + \tfrac{1}{2}mc|d\vec{\beta}|^2$$

and a spacelike part

$$\vec{0} = \vec{\beta}_1 \frac{dE_1}{c} + mc\, d\vec{\beta} \ .$$

In the timelike equation the mc terms cancel, and the last term is negligible because it is second order in the small quantity $d\beta$. This equation reduces to $dE_1 = c^2 dm$. Substituting for dE_1 in the spacelike equation, and taking account of the fact that $\vec{\beta}_1$ and $d\vec{\beta}$ point in opposite directions, we obtain

$$|d\vec{\beta}| = |\vec{\beta}_1| \frac{|dm|}{m} \ .$$

When a second particle is ejected, we set up a different comoving frame and compute an analogous $|d\vec{\beta}|$. As in section 11, our difficulty is that the two $|d\vec{\beta}|$'s don't add. What *do* add are the two $d\eta$'s; fortunately, since the rocket moves nonrelativistically relative to the comoving frame, we can easily equate $d\eta \approx |d\vec{\beta}|$. Therefore, summing over the emission of many particles,

$$\begin{aligned} \eta_{\text{final}} - (\eta_0 \equiv 0) &= \int_{m_0}^{m_{\text{final}}} |\vec{\beta}_1| \frac{|dm|}{m} \\ \eta_{\text{final}} &= |\vec{\beta}_1| \ln \frac{m_0}{m_{\text{final}}} \ . \end{aligned} \tag{31}$$

This is the classic *rocket equation*: the achievable Lorentz boost increases linearly with the relative exhaust velocity $\beta_1 c$, but only logarithmically with the ratio of initial to final rocket masses.

Chemical rocket engines achieve a maximum $|\vec{\beta}_1| \approx 4 \times 10^3$ m/sec/$c \approx 1.33 \times 10^{-5}$. To achieve a boost $\eta_{\text{final}} = 10.34$ as in section 11, we would require

$$\ln \frac{m_0}{m_{\text{final}}} = 7.8 \times 10^5 \ ,$$

yielding a mass ratio that is beyond calculator range. Evidently chemical rockets (the only type used up to now) will never suffice.

Relativistic rocket engines emit particles at $\beta_1 \approx 1$. If they were perfectly efficient,

$$\ln \frac{m_0}{m_{\rm final}} = 10.34$$
$$m_0 = 3.1 \times 10^4 m_{\rm final} \ .$$

If the rocket were to carry a payload that includes an astronaut, $m_{\rm final} > 10$ T would be needed to provide life support. Then the initial rocket mass would be

$$m_0 > 3.1 \times 10^5 \ \mathrm{T} \ ,$$

heavier than an aircraft carrier. Note that Eq. (31) becomes

$$\eta_{\rm final} = \epsilon \, |\vec{\beta}_1| \, \ln \frac{m_0}{m_{\rm final}}$$

if the efficiency ϵ of the engine is less than unity.

Unfortunately, present relativistic rocket engine concepts are grossly inefficient ($\epsilon \ll 1$), and leave most of their fuel on board so that $m_0/m_{\rm final}$ cannot be $\gg 1$. A simple example is a laser powered by batteries. Much engineering remains to be accomplished, even for the modest goal of propelling astronauts through only an infinitesimal fraction of the universe.

15. The four-gradient

So far we have discussed two four-vectors: $r \equiv (ct, \vec{r})$ and $p \equiv (E/c, \vec{p})$, where $E \equiv \gamma mc^2$ and $\vec{p} \equiv \gamma m\vec{v}$. We'll briefly mention some other four-vectors here and in sections 16 and 17.

Using standard methods of differential calculus, starting from the Lorentz transformation law for r, it is straightforward to show that

$$\begin{pmatrix} \frac{\partial}{c\partial t'} \\ -\frac{\partial}{\partial x'} \\ -\frac{\partial}{\partial y'} \\ -\frac{\partial}{\partial z'} \end{pmatrix} = \begin{pmatrix} \gamma_0 & -\gamma_0\beta_0 & 0 & 0 \\ -\gamma_0\beta_0 & \gamma_0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{c\partial t} \\ -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{pmatrix}$$

That is, the *four-gradient* operator

$$\partial \equiv \begin{pmatrix} \frac{\partial}{c\partial t} \\ -\frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} \\ -\frac{\partial}{\partial z} \end{pmatrix} \equiv \left(\frac{\partial}{c\partial t}, -\vec{\nabla}\right) \qquad (32)$$

transforms like a four-vector as well. (Note the minus sign in front of $\vec{\nabla}$; because of it, $\partial \cdot r = 4$, not -2.)

The EM *wave equation operator*

$$\begin{aligned} \partial \cdot \partial &\equiv \frac{\partial^2}{c^2\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \\ &\equiv \frac{\partial^2}{c^2\partial t^2} - \nabla^2 \end{aligned} \qquad (33)$$

is the inner product of two four-vectors, and therefore is a Lorentz invariant.

16. Electromagnetic four-vectors

The *continuity equation* that enforces charge conservation is often written

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \qquad (34)$$

where ρ is the volume charge density (Coul/m^3) and $\vec{J}$ is the volume current density (Amp/m^2). Equation (34) is equivalent to

$$\partial \cdot J = 0 \ ,$$

where

$$J \equiv (c\rho, \vec{J}) \qquad (35)$$

is the *four-current density*. Because ∂ is a four-vector and $\partial \cdot J$ is a Lorentz invariant, J must transform like ∂ and therefore must also be a four-vector.

Both of the sourceless Maxwell equations

$$\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \end{aligned} \qquad (36)$$

are implicit in the relations

$$\begin{aligned} \vec{B} &\equiv \vec{\nabla} \times \vec{A} \\ \vec{E} &\equiv -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \ , \end{aligned} \qquad (37)$$

where Φ is the *scalar potential* and $\vec{A}$ is the *vector potential*. All potentials have some freedom in their definition; for example, the potential energy associated with a mechanical problem can be modified by an additive constant without changing the motion. The freedom enjoyed by the electromagnetic potentials is called *gauge invariance*. It turns out that, because of gauge invariance, we are free to impose upon Φ and $\vec{A}$ the *Lorentz gauge condition*

$$\frac{1}{c^2}\frac{\partial\Phi}{\partial t} + \vec{\nabla}\cdot\vec{A} = 0 \tag{38}$$

Defining the *four-potential*

$$A \equiv (\frac{\Phi}{c}, \vec{A}) \ , \tag{39}$$

Eq. (38) can be rewritten

$$\partial \cdot A = 0 \ .$$

Because $\partial \cdot A$ is a Lorentz invariant, and ∂ is a four-vector, the four-potential A must also be a four-vector.

When the Lorentz gauge condition is imposed, it turns out that both of the sourceful Maxwell equations

$$\begin{aligned} \nabla\cdot\vec{E} &= \frac{\rho}{\epsilon_0} \\ \nabla\times\vec{B} &= \mu_0\vec{J} + \epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{40}$$

can be rewritten as the single four-vector equation

$$(\partial\cdot\partial)A = \mu_0 J \tag{41}$$

That is, the EM wave equation operator acts upon the four-potential A to yield the four-current J ($\times\mu_0$ in SI units). Since the two sourceless Maxwell equations are implicit in the definition of A, Eq. (41) carries as much information as all four of Maxwell's!

Knowing how A transforms and how the electromagnetic fields are derived from it, with some algebra we can deduce how $\vec{E}$ and $\vec{B}$ themselves transform. The result is:

$$\begin{aligned} \vec{E}'_\perp &= \gamma_0(\vec{E}_\perp + \vec{\beta}_0\times c\vec{B}_\perp) \\ c\vec{B}'_\perp &= \gamma_0(c\vec{B}_\perp - \vec{\beta}_0\times\vec{E}_\perp) \\ E'_\parallel &= E_\parallel \\ cB'_\parallel &= cB_\parallel \ , \end{aligned} \tag{42}$$

where "$_\parallel$" refers to the coordinate $\hat{\beta}_0$ along which $\mathcal{S}'$ is moving relative to $\mathcal{S}$ ($= \hat{x}$ in earlier examples), and "$_\perp$" refers to any direction perpendicular to that coordinate.

17. The wave four-vector

Suppose that you run toward me; at a certain time you begin to emit waves (of any kind). By the time we collide, I will have felt all N wave maxima that you emitted. Therefore we both must agree on the accumulated phase $2\pi N$ of that wave; that phase must be a Lorentz invariant.

A plane wave travelling in the $\hat{x}$ direction can be considered to be a function of $\omega t - k_x x$, where ω is the wave's *angular frequency* and k_x is the $\hat{x}$ component of its *wave vector* $\vec{k}$. The wave's *phase velocity* $v_{\rm ph}$ is

$$v_{\rm ph} = \frac{\omega}{|\vec{k}|} \ . \tag{43}$$

More generally, for an arbitrary direction of propagation $\hat{k}$, the wave's phase is

$$\omega t - \vec{k}\cdot\vec{r} \equiv k\cdot r \ , \tag{44}$$

where the *wave four-vector* k is defined as

$$k \equiv (\frac{\omega}{c}, \vec{k}) \ . \tag{45}$$

Since the phase is a Lorentz invariant and r is a four-vector, k must also be a four-vector.

18. Relativistic Doppler shift

Figure 10 shows a wave source at rest in $\mathcal{S}'$ and an observer at rest in $\mathcal{S}$. What relates the angular frequencies ω' and ω with which the wave is emitted and observed?

Applying the Lorentz transformation to the zeroth component of the wave four-vector k,

$$\frac{\omega'}{c} = \gamma_0\frac{\omega}{c} - \gamma_0\beta_0 k_x \ . \tag{46}$$

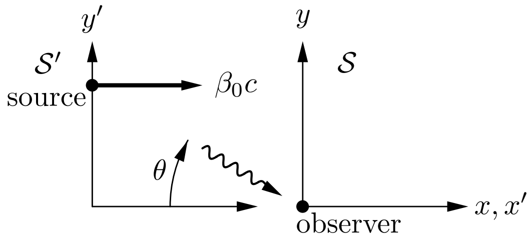

FIG. 10. Geometry for analysis of relativistic Doppler shift. Lab frame $\mathcal{S}$ and source frame $\mathcal{S}'$ are arranged as in Fig. 2. As detected by an observer at rest at the origin of lab frame $\mathcal{S}$, the wave has angular frequency ω, phase velocity $\beta_{\rm ph}c$, and angle θ with respect to the $\hat{x} = \hat{x}'$ direction.

Let the phase velocity of the wave as observed in $\mathcal{S}$ be $\beta_{\rm ph}c$ ($\beta_{\rm ph} = 1$ for a light wave). From Eq. (43),

$$|\vec{k}| = \frac{\omega}{v_{\rm ph}} = \frac{\omega}{\beta_{\rm ph}c}$$
$$k_x = \frac{\omega}{\beta_{\rm ph}c}\cos\theta\ ,$$

where θ is the wave's angle with respect to the direction of $\mathcal{S}$'s motion relative to $\mathcal{S}$. Plugging k_x into Eq. (46),

$$\frac{\omega'}{c} = \gamma_0\frac{\omega}{c} - \gamma_0\beta_0\frac{\omega}{\beta_{\rm ph}c}$$
$$\omega = \frac{\omega'}{\gamma_0\left(1 - \frac{\beta_0}{\beta_{\rm ph}}\cos\theta\right)}\ . \qquad (47)$$

Equation (47) describes the *relativistic Doppler shift.*

A singularity occurs when

$$\cos\theta = \frac{\beta_{\rm ph}}{\beta_0} = \frac{v_{\rm ph}}{V}$$

(obviously possible only when $V > v_{\rm ph}$). When V describes a speedboat and $v_{\rm ph}$ describes a water-surface wave, this singularity is called a *bow wave*; when V describes a jet and $v_{\rm ph}$ describes a sound wave, it is called a *sonic boom.*

FIG. 11. Sonic boom.

When V describes a relativistic particle travelling through transparent material and $V_{\rm ph}$ describes light propagating through that same material, the singularity is called *Cherenkov radiation.*

As noted above, when the wave is a light wave propagating in vacuum, $\beta_{\rm ph} = 1$. In that special case, Eq. (47) becomes

$$\omega = \frac{\omega'}{\gamma_0(1 - \beta_0\cos\theta)}\ .$$

Further, if the light wave is approaching or receding head-on,

$$\omega_{\rm recede}^{\rm approach} = \frac{\omega'}{\gamma_0(1 \mp \beta_0)} = \sqrt{\frac{1 \pm \beta_0}{1 \mp \beta_0}}\,\omega'\ .$$

Alternatively, if the light wave is incident from the zenith ($\cos\theta = 0$), where nonrelativistically there would be no Doppler shift,

$$\omega = \frac{\omega'}{\gamma_0}\quad \text{(ordinary time dilation).}$$

Nonrelativistically ($\beta_0 \ll 1$),

$$\omega = \frac{\omega'}{\left(1 - \frac{V}{v_{\rm ph}} \cos\theta\right)} \ .$$

This last equation (sometimes further restricted to $\theta = 0$ or π) is the Doppler formula found in freshman texts.

19. Aberration

An analysis similar to that for the relativistic Doppler shift yields the relativistic relations for *aberration* of a wave

$$\tan\theta' = \frac{\sin\theta}{\gamma_0(\cos\theta - \beta_0\beta_{\rm ph})} \ , \tag{48}$$

and of a particle

$$\tan\theta' = \frac{\sin\theta}{\gamma_0(\cos\theta - \beta_0/\beta)} \ , \tag{49}$$

where, for the particle, βc is its laboratory velocity. Again, θ is the angle (relative to $\vec{\beta}_0$) of the wave or particle as observed in the lab frame $\mathcal{S}$, and θ' is the same angle as observed in the source frame $\mathcal{S}'$.

20. Covariant notation

So far, we have chosen to use an "intuitive" notation for the components of common four-vectors. For example, our notation for the spacetime and energy-momentum four-vectors has been

$$r \equiv (ct, \vec{r}) \equiv (ct, x, y, z)$$
$$p \equiv (\tfrac{E}{c}, \vec{p}) \equiv (\tfrac{E}{c}, p_x, p_y, p_z) \ .$$

This choice has served us well– it constantly has reminded us of the physical meaning of each four-vector component. Now we venture beyond this "intuitive" notation because we need to write formulæ that take sums over four-vector components. These formulæ can be written much more elegantly if the components are labeled by an integer subscript or superscript. Following established convention, we will use *super*scripts to denote the components of standard ("contravariant") four-vectors.

Conventionally, a *three*-vector has components labeled by integer indices running from 1 to 3; when such indices are exhibited symbolically, *italic* letters are used. For example, the three-vector $\vec{W}$, where W is a randomly chosen letter having no specific meaning, up to now has been denoted by $\vec{W} \equiv (W_x, W_y, W_z)$. However, in our new covariant notation, $\vec{W}$ is denoted by

$$\vec{W} \equiv (W^1, W^2, W^3) \ .$$

Its magnitude^2 is denoted by

$$\begin{aligned} |\vec{W}|^2 &\equiv \vec{W} \cdot \vec{W} \\ &= W^1W^1 + W^2W^2 + W^3W^3 \\ &= \sum_{i=1}^{3} W^iW^i \\ &\equiv W^iW^i \ . \end{aligned}$$

Note that the dot product of $\vec{W}$ with itself is now easily expressed as a sum over the integer index i. In the last line, we have made use of an additional convention: *repeated indices are summed* over their natural range, in this case 1 to 3.

By the same convention, a *four*-vector has components labeled by integer indices running from *0* to 3; when such indices are exhibited symbolically, *Greek* letters are used. For example, in covariant notation, the four-vector W is denoted by

$$W \equiv (W^0, W^1, W^2, W^3) \ .$$

An individual component of W is denoted, for example, by W^μ, where μ can take on the values 0, 1, 2, or 3. Its magnitude^2 is denoted by

$$W \cdot W = W^0W^0 - W^1W^1 - W^2W^2 - W^3W^3 \ .$$

[Now, it may be the case that the $0^{\rm th}$ component of a four-vector would more intuitively be denoted by a different symbol; for example, calling the spacetime four-vector x, it would more natural to write ct rather than x^0 as its $0^{\rm th}$ component. Nevertheless, to take advantage of covariant notation, we must write the $0^{\rm th}$ component of x as x^0 and we must remember that it is equivalent to ct.]

A four-*tensor* has components labeled by two Greek indices. For example, one of the sixteen components of the four-tensor $\mathcal{W}$ is denoted by $\mathcal{W}^{\mu\nu}$, where μ and ν independently take on integer values ranging from 0 to 3. When $\mathcal{W}$ is written as a 4×4 matrix, its first index μ is the row index; ν is the column index.

21. Transformations in covariant notation

First we consider 3D spatial rotations. Denote by x^i the i^{th} component of the position vector $\vec{x}$, and denote by λ^i_j the component in the i^{th} row and j^{th} column of the 3×3 submatrix of Λ_R [defined after Eq. (21)]. The real 3D rotation matrix λ must be *orthogonal* ($\lambda^t = \lambda^{-1}$) and, if the rotation is *proper*, λ must be obtainable as the result of multiplying the identity matrix by an infinite number of matrix operators corresponding to infinitesimal rotations; this requires $\det \lambda = +1$. For example, if λ represents a passive rotation by counterclockwise angle ϕ about the z axis, as in Eq. (9),

$$\lambda = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The components of $\vec{x}\,'$, the vector as it appears in the rotated coordinate system $\mathcal{S}'$, are given by

$$(x')^i = \lambda^i{}_j x^j \; . \tag{50}$$

In covariant notation, this is the 3D analog of Eq. (9).

Denote by T^{ij} the element in the i^{th} row and j^{th} column of a three-tensor (for example, a rigid body's inertia tensor). Under the passive rotation λ, the components of T', the tensor as it appears in $\mathcal{S}'$, are given by

$$(T')^{ij} = \lambda^i{}_k \lambda^j{}_l T^{kl} \; . \tag{51}$$

More precisely, the fact that T transforms according to Eq. (51) *defines* it to be a three-tensor.

Now we graduate to 4D spacetime (Lorentz) transformations. Denote by x^μ the μ^{th} component of the spacetime vector x, and denote by $\Lambda^\mu{}_\nu$ the component in the μ^{th} row and ν^{th} column of the Lorentz transformation matrix Λ [cf. Eqs. (19) and (20)]. Λ is a real symmetric 4×4 matrix with unit determinant; it satisfies the condition $\Lambda(\vec{\beta}_0) = \Lambda^{-1}(-\vec{\beta}_0)$. The components of x', the spacetime vector as it appears in the moving coordinate system $\mathcal{S}'$, are given by

$$(x')^\mu = \Lambda^\mu{}_\nu x^\nu \; . \tag{52}$$

In covariant notation, this is the analog of Eq. (20).

Denote by $F^{\mu\nu}$ the element in the μ^{th} row and ν^{th} column of a Lorentz four-tensor (examples to follow). Under the Lorentz tranformation Λ, the components of F', the four-tensor as it appears in $\mathcal{S}'$, are given similarly by

$$(F')^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma} \; . \tag{53}$$

Note that the RHS of this equation has 16 terms.

22. Metric tensor

What's required in order to write dot products in covariant notation? In analogy to the three-vector dot product, $\vec{x} \cdot \vec{x} = x^i x^i$, naïvely we might write the four-vector dot product as $x \cdot x = x^\mu x^\mu$. But this would make all four terms positive, while we know that the last three must be negative. To remedy this error we introduce the *metric tensor* g, where

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

with elements denoted by $g_{\mu\nu}$. Using g we can write

$$x \cdot x = x^\mu g_{\mu\nu} x^\nu \; . \tag{54}$$

In covariant notation, this is the analog of Eq. (2). Note that (like Λ) the metric tensor is not itself a Lorentz four-tensor; g retains the same elements in any Lorentz frame. The metric tensor is the gateway to *general relativity*; in curved spacetime its off-diagonal elements no longer vanish.

In 3D, the dot product of a three-tensor T and a three-vector $\vec{a}$ is another three-vector $\vec{b}$:

$$\begin{aligned} T \cdot \vec{a} &= \vec{b} \\ T^{ij} a^j &= b^i \ . \end{aligned} \tag{55}$$

Likewise, in 4D, the dot product of a Lorentz four-tensor F and a four-vector a is another four-vector b. Here, as with the dot product of two four-vectors, we must incorporate the metric tensor:

$$\begin{aligned} F \cdot a &= b \\ F^{\mu\nu} g_{\nu\rho} a^\rho &= b^\mu \ . \end{aligned} \tag{56}$$

As a rule of thumb in typical formulæ, you can tell where metric tensor elements are needed by noting the heights (superscript or subscript) of the repeated Greek indices that are to be summed. You keep inserting elements of the metric tensor until each pair of repeated indices includes one superscript and one subscript.

23. Covariant *vs.* contravariant 4-vectors

In the wake of Eqs. (54) and (56), you seem destined to write lots of $g_{\mu\nu}$'s in your career. But you are rescued by the shorthand made possible by another convention. Considering the four-vector x, define

$$x_\mu \equiv g_{\mu\nu} x^\nu = x^\nu g_{\nu\mu} \ . \tag{57}$$

If x^μ is an element of (a, b, c, d), x_μ is an element of $(a, -b, -c, -d)$: the timelike element is the same, but the spacelike elements differ by a sign. The four-vector whose components are x_μ is called the *covariant* form of the (standardly *contravariant*) four-vector whose components are x^μ. Covariant four-vectors Lorentz transform differently than standard (contravariant) four-vectors:

$$(x')_\mu = x_\nu (\Lambda^{-1})^\nu{}_\mu \ . \tag{52a}$$

This allows $x_\mu x^\mu$ to be a Lorentz scalar.

Likewise, considering the four-tensor F, define

$$\begin{aligned} F^\mu{}_\nu &\equiv F^{\mu\rho} g_{\rho\nu} \\ F_\mu{}^\nu &\equiv g_{\mu\rho} F^{\rho\nu} \\ F_{\mu\nu} &\equiv g_{\mu\rho} F^{\rho\sigma} g_{\sigma\nu} \ . \end{aligned} \tag{58}$$

If $F^{\mu\nu}$ is an element of

$$\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} ,$$

then $F^\mu{}_\nu$, $F_\mu{}^\nu$, and $F_{\mu\nu}$ are elements, respectively, of

$$\begin{pmatrix} a & -b & -c & -d \\ e & -f & -g & -h \\ i & -j & -k & -l \\ m & -n & -o & -p \end{pmatrix} ,$$

$$\begin{pmatrix} a & b & c & d \\ -e & -f & -g- & -h \\ -i & -j & -k & -l \\ -m & -n & -o & -p \end{pmatrix} ,$$

$$\text{and} \quad \begin{pmatrix} a & -b & -c & -d \\ -e & f & g & h \\ -i & j & k & l \\ -m & n & o & p \end{pmatrix} .$$

The semi- or fully-covariant forms of four-tensors likewise Lorentz transform differently from the standard (fully contravariant) form. For example,

$$(F')_{\mu\nu} = F_{\rho\sigma} (\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu \ . \tag{53a}$$

This allows $F_{\mu\nu} F^{\mu\nu}$ to be a Lorentz scalar.

Using covariant notation, you can eliminate the $g_{\mu\nu}$'s from your equations by changing the appropriate indices from superscripts to subscripts. Again, pairs of repeated indices that are to be summed should consist of one superscript and one subscript.

Taking advantage of this covariant shorthand, Eqs. (54) and (56) simplify to

$$x \cdot x = x_\mu x^\mu \tag{54$'$}$$

$$F^{\mu\nu} a_\nu \ \text{ or } \ F^\mu{}_\nu a^\nu = b^\mu \ . \tag{56$'$}$$

24. Manifestly covariant equations

Before introducing more new formalism, let's step back and exploit what we've learned to rewrite some of the fundamental equations of physics in *manifestly covariant* form. Now, if any equation is correct, the Lorentz transformation properties of the left-hand side (LHS) and RHS must be the same. But when the equation's form is manifestly covariant, it's *totally obvious* that the LHS and RHS transform in the same way – they are obviously both Lorentz scalars, or elements of four-vectors, *etc.* To the eye of the relativistically oriented physicist, equations are most elegant when written in manifestly covariant form.

We'll start by rewriting equations that have already appeared in these notes.

Definition of four-momentum:

$$p^\mu \equiv m\,\frac{dx^\mu}{d\tau} \qquad (25')$$

Definition of the contravariant derivative...

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} \qquad (32')$$

... and of the covariant derivative:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad (59)$$

Charge conservation (continuity equation):

$$\partial_\mu J^\mu = 0 \qquad (34')$$

Lorentz gauge condition:

$$\partial_\mu A^\mu = 0 \qquad (38')$$

Sourceful Maxwell equations in Lorentz gauge:

$$(\partial_\mu \partial^\mu) A^\nu = \mu_0 J^\nu \qquad (41')$$

We add the *generalized de Broglie relation*:

$$p^\mu = \hbar k^\mu \qquad (60)$$

Note that this relation includes not only the usual $\lambda = h/|\vec{p}|$ but also Planck's relation $E = h\nu$.

25. Manifestly covariant Maxwell's equations

On the MIT campus in the 1960's, during the Sputnik-induced US science boom, students wore tee shirts festooned with Maxwell's equations. Very temporarily embracing that fashion sense, today we ask what the least inelegant such tee shirt might have read.

Surely the freshman version

$$\oint \vec{B}\cdot d\vec{a} = 0$$
$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$
$$(\mathcal{E} \equiv \oint \vec{E}\cdot d\vec{\ell})$$
$$(\Phi_B \equiv \int \vec{B}\cdot d\vec{a})$$

$$\oint \epsilon_0 \vec{E}\cdot d\vec{a} = Q$$
$$(Q \equiv \oint \rho\, d\tau)$$
$$\oint \tfrac{1}{\mu_0}\vec{B}\cdot d\vec{\ell} = I + I_d$$
$$(I \equiv \int \vec{J}\cdot d\vec{a})$$
$$(I_d \equiv \int \epsilon_0 \vec{E}\cdot d\vec{a})$$

wins no prize. (Note that we have separated the first two sourceless equations from the last two "sourceful" ones.)

The sophomore version of Maxwell's equations is slightly less geeky:

$$\nabla\cdot\vec{B} = 0$$
$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

$$\nabla\cdot\epsilon_0\vec{E} = \rho$$
$$\nabla\times\tfrac{1}{\mu_0}\vec{B} = \vec{J} + \frac{\partial}{\partial t}\epsilon_0\vec{E}\ .$$

The junior version takes advantage of the scalar and vector EM potentials; thanks to Eqs. (36)

and (37), the sourceless Maxwell equations are satisfied automatically. We are left with

$$(\partial_\mu \partial^\mu) A^\nu = \mu_0 J^\nu$$
$$\left(\partial_\mu A^\mu = 0\right) ,$$

where the second equation reminds us that the first is valid only if the Lorentz gauge condition is imposed. These equations are manifestly covarant; the tee shirt sporting them is an improved fashion statement.

Can the MIT "tech tool" do better? If he is to write only one equation, it must be valid for any choice of EM gauge. Define the *field strength tensor*

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu \ . \tag{61}$$

Because it is constructed out of two four-vectors, F must transform according to Eq. (53); therefore it is a Lorentz four-tensor. Because it is antisymmetric, F has only six independent elements; it is straightforward to show that they are proportional to the three components of $\vec{E}/c$ and $\vec{B}$:

$$F = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix} \tag{62}$$

The covariant four-derivative ∂_μ, when it operates on $F^{\mu\nu}$, must yield a four-vector; this is nothing more than $\mu_0 \times$ the four-current-density J^ν. So the winning tee shirt proclaims

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \ . \tag{63}$$

Explicitly or implicitly, Eq. (63) includes the information in all four of Maxwell's equations, and it is valid in any EM gauge.

26. Lorentz transformation of EM fields

Equations (42) gave a prescription for Lorentz transforming the EM fields $\vec{E}$ and $\vec{B}$; it was claimed that this prescription follows from the rules for transforming the EM four-potential A^μ "after some algebra". Just what is this algebra?

The most straightforward derivation of Eqs. (42) begins with the field strength tensor $F^{\mu\nu}$ and the rule (Eq. (53)) for transforming it. Using this approach, the algebra is manageable.

After Eq. (53a) we stated that $F_{\mu\nu}F^{\mu\nu}$, where F is any Lorentz four-tensor, is a Lorentz scalar. (This is true also for $F_{\mu\nu}G^{\mu\nu}$, where F and G are any two Lorentz four-tensors.) When F is the field strength tensor, to what is the Lorentz scalar $F_{\mu\nu}F^{\mu\nu}$ equal? It is easy to show that

$$F_{\mu\nu}F^{\mu\nu} = -\tfrac{2}{c^2}(|\vec{E}|^2 - |c\vec{B}|^2) \ .$$

Therefore $|\vec{E}|^2 - |c\vec{B}|^2$ has the same value in any Lorentz frame. [With less elegance this can be deduced directly from Eqs. (42).]

Can we form a second Lorentz scalar from $\vec{E}$ and $\vec{B}$? Our approach is first to identify another Lorentz four-tensor G whose elements are functions only of $\vec{E}$ and $\vec{B}$, and then to examine the Lorentz scalars $G_{\mu\nu}G^{\mu\nu}$ and $F_{\mu\nu}G^{\mu\nu}$.

The *dual field strength tensor* G is defined by

$$G^{\mu\nu} \equiv \tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \ , \tag{64}$$

where $\epsilon^{\mu\nu\rho\sigma} \equiv 1$ (-1) when $\mu\nu\rho\sigma$ is an even (odd) permutation of 0123, and 0 otherwise. G is a Lorentz four-tensor, while g is a fourth-rank four-tensor whose elements retain the same values in any Lorentz frame. It is straightforward to evaluate

$$G = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3/c & -E^2/c \\ B^2 & -E^3/c & 0 & E^1/c \\ B^3 & E^2/c & -E^1/c & 0 \end{pmatrix} \tag{65}$$

Notice that G is obtained from the standard field strength tensor F by changing $\vec{E}/c \to \vec{B}$ and $\vec{B} \to -\vec{E}/c$.

Unfortunately, $G_{\mu\nu}G^{\mu\nu}$ yields no new Lorentz scalar; again it is proportional to $|\vec{E}|^2 - |c\vec{B}|^2$. However, $F_{\mu\nu}G^{\mu\nu}$ is easily shown to be proportional to $\vec{E} \cdot \vec{B}$. Therefore $\vec{E} \cdot \vec{B}$ is our second Lorentz scalar. Again, this may be deduced less elegantly from Eqs. (42).

We collect a bonus from introducing the dual field strength tensor. Earlier, we remarked that the sourceless Maxwell equations are implicit in the definition of the EM four-potential A^μ. But if you insist on an explicit, manifestly covariant statement of these two equations, it is simply

$$\partial_\mu G^{\mu\nu} = 0 \ . \qquad (66)$$

Perhaps Eq. (66) should be added to Eq. (63) on the winning tee shirt.

27. Motion of a charged particle in uniform static EM fields

Under the influence of an EM field, a point particle of charge e and rest mass m moves according to the Lorentz force law:

$$\frac{d\vec{p}}{dt} = e(\vec{E} + \vec{\beta} \times c\vec{B}) \ , \qquad (67)$$

where $\vec{p}$ is the *relativistically correct* momentum, *i.e.* the spacelike component of the energy-momentum four-vector:

$$\vec{p} \equiv \gamma\vec{\beta}mc \ , \qquad \gamma \equiv \frac{1}{\sqrt{1-|\vec{\beta}|^2}} \ , \qquad (68)$$

with $\vec{\beta}c$ denoting the particle's velocity.

(Case I) In a uniform static *magnetic* field $\vec{B}_0$, with $\vec{E} = 0$, a straightforward consequence of Eq. (67) is that the particle executes helical motion about $\hat{B}_0$ with angular frequency

$$\omega_{\rm cyc} = \frac{e|\vec{B}_0|}{\gamma m} \ , \qquad (69)$$

where $|\vec{\beta}|$ and γ do not vary. For a nonrelativistic proton with $\gamma = 1$, the *cyclotron frequency* $\omega_{\rm cyc} = 95.6$ MHz/Tesla. The radius of the helix is

$$R = \frac{|\vec{p}_\perp|}{e|\vec{B}_0|} \ ,$$

where $\vec{p}_\perp$ is the particle's momentum transverse to $\hat{B}_0$ (again $|\vec{p}_\perp|$ does not vary).

(Case II) Conversely, when the particle moves in a uniform static *electric* field $\vec{E}_0$, with $\vec{B} = 0$, the component $p_\parallel = \gamma\beta_\parallel mc$ of its momentum parallel to $\hat{E}_0$ changes linearly with time according to

$$\frac{dp_\parallel}{dt} = \frac{e|\vec{E}_0|}{m} \ ,$$

while $\vec{p}_\perp$ remains fixed. The particle's trajectory is a hyperbola (which approaches the usual parabola in the nonrelativistic limit $\beta \ll 1$).

(Case III) When both $\vec{B}_0$ and $\vec{E}_0$ are nonzero, the motion can be complicated. However, case III can be reduced to the simpler cases I or II if the problem can be worked in a Lorentz frame $\mathcal{S}'$ in which either $\vec{E}'_0$ or $\vec{B}'_0$ vanish. Can such a frame be found?

Here's where the Lorentz invariants $\vec{E} \cdot \vec{B}$ and $|\vec{E}|^2 - |c\vec{B}|^2$ have practical impact. Unless $\vec{E}_0 \cdot \vec{B}_0 = 0$ in the problem as originally posed, $\vec{E}'_0 \cdot \vec{B}'_0$ will be nonzero in any $\mathcal{S}'$, which will force both $\vec{E}'_0$ and $\vec{B}'_0$ not to vanish. So one of the fields can be transformed away only if $\vec{E}_0 \perp \vec{B}_0$.

In $\mathcal{S}'$, which field can be eliminated? If $|\vec{E}_0|^2 - |c\vec{B}_0|^2) < 0$, $\vec{B}'_0$ will survive, and case I will obtain. If $|\vec{E}_0|^2 - |c\vec{B}_0|^2) > 0$, $\vec{E}'_0$ will survive, yielding case II. If $|\vec{E}_0|^2 = |c\vec{B}_0|^2$, a frame does exist in which both fields vanish, but it cannot be reached by a Lorentz transformation with finite boost.

More reading

The following table suggests references to [G] Griffiths, *Introduction to Electrodynamics*, 3rd ed.; [TW] Taylor & Wheeler, *Spacetime Physics*; and [J] Jackson, *Classical Electrodynamics*, 3rd ed.; organized by this course's 27 topics.

Topic	**Reference**
1-6	TW 1-9
7	G 12.1.4
8-9	G 12.1.2
10	TW 9
11	TW Ex. 51
12-13	G 12.2.1-3
14	TW Ex. 58
15	G Problem 12.55
16	G 12.3.4
17-19	J 11.3D
20-21	G 12.1.4
22-23	J 11.6
24-26	G 12.3.3-5; J 11.9
27	G 12.2.4